

# Quantified separably injective spaces

Duanxu Dai<sup>†</sup>

School of Mathematical Sciences

Xiamen University

Xiamen 361005, China

E-mail: dduanxu@163.com

Present address:

Department of Mathematics

Texas A&M University

College Station, TX 77843-3368, USA

## Abstract

Let  $X, Y$  be two Banach spaces. Let  $\varepsilon \geq 0$ . A mapping  $f : X \rightarrow Y$  is said to be a standard  $\varepsilon$ -isometry if  $f(0) = 0$  and  $|||f(x) - f(y)|| - ||x - y||| \leq \varepsilon$ . In this paper we first show that if  $Y^*$  has the point of  $w^*$ -norm continuity property (in short,  $w^*$ -PCP) or  $Y$  is separable, then for every standard  $\varepsilon$ -isometry  $f : X \rightarrow Y$  there exists a  $w^*$ -dense  $G_\delta$  subset  $\Omega$  of  $ExtB_{X^*}$  such that there is a bounded linear operator  $T : Y \rightarrow C(\Omega, \tau_{w^*})$  with  $\|T\| = 1$  such that  $Tf - Id$  is uniformly bounded by  $4\varepsilon$  on  $X$ . As a corollary we obtain quantitative characterizations of separably injectivity of a Banach space and its dual that turn out to give a positive answer to Qian's problem of 1995 in the setting of universality. We also discuss Qian's problem for  $\mathcal{L}_{\infty, \lambda}$ -spaces and  $C(K)$ -spaces. Finally, we prove a sharpen quantitative and generalized Sobczyk theorem.

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2010 *Mathematics Subject Classification*: Primary 46B04, 46B20, 47A58; Secondary 26E25, 54C60, 54C65, 46A20.

*Key words and phrases*:  $\varepsilon$ -Isometry, Stability, Point of continuity property, Injective Banach space, Separably injective Banach space.

<sup>†</sup> Supported in part by a fund of China Scholarship Council and Texas A&M University.

# 1 Introduction

That every surjective isometry between two Banach spaces  $X$  and  $Y$  is necessarily affine was proved by Mazur and Ulam [26] in 1932. Since then, properties of isometries and generalizations thereof between Banach spaces has continued for 82 years. On this period, many significant problems about perturbation properties of surjective  $\varepsilon$ -isometries were proposed and solved by numerous mathematicians. In particular, we mention the Hyers-Ulam problem [21] (see, for instance, [17], [19], and [27]). In 1968, Figiel [16] showed the following remarkable result: For every standard isometry  $f : X \rightarrow Y$  there is a linear operator  $T : L(f) \rightarrow X$  with  $\|T\| = 1$  so that  $Tf = Id$  on  $X$ , where  $L(f)$  is the closure of  $\text{span } f(X)$  in  $Y$  (see also [7] and [14]). In 2003, Godefroy and Kalton [18] studied the relationship between isometries and linear isometries and solved a long-standing problem: Does the existence of an isometry  $f : X \rightarrow Y$  imply the existence of a linear isometry  $U : X \rightarrow Y$ ?

**Definition 1.1.** Let  $X, Y$  be two Banach spaces,  $\varepsilon \geq 0$ , and let  $f : X \rightarrow Y$  be a mapping.

(1)  $f$  is said to be an  $\varepsilon$ -isometry if

$$(1.1) \quad |||f(x) - f(y)|| - \|x - y|| \leq \varepsilon \text{ for all } x, y \in X.$$

In particular, a 0-isometry  $f$  is simply called an isometry.

(2) We say an  $\varepsilon$ -isometry  $f$  is standard if  $f(0) = 0$ .

(3) A standard  $\varepsilon$ -isometry  $f$  is  $(\alpha, \gamma)$ -stable if there exist  $\alpha, \gamma > 0$  and a bounded linear operator  $T : L(f) \rightarrow X$  with  $\|T\| \leq \alpha$  such that

$$(1.2) \quad \|Tf(x) - x\| \leq \gamma\varepsilon, \text{ for all } x \in X.$$

In this case, we also simply say  $f$  is stable, if no confusion arises.

(4) A pair  $(X, Y)$  of Banach spaces  $X$  and  $Y$  is said to be stable if every standard  $\varepsilon$ -isometry  $f : X \rightarrow Y$  is  $(\alpha, \gamma)$ -stable for some  $\alpha, \gamma > 0$ .

(5) A pair  $(X, Y)$  of Banach spaces  $X$  and  $Y$  is called  $(\alpha, \gamma)$ -stable for some  $\alpha, \gamma > 0$  if every standard  $\varepsilon$ -isometry  $f : X \rightarrow Y$  is  $(\alpha, \gamma)$ -stable.

The study of non-surjective  $\varepsilon$ -isometries has also been considered (see, for instance, [5], [10], [11], [12], [13], [27], [30], [32] and [34]). Qian[30] proposed the following problem in 1995.

**Problem 1.2.** *Is it true that for every pair  $(X, Y)$  of Banach spaces  $X$  and  $Y$  there exists  $\gamma > 0$  such that every standard  $\varepsilon$ -isometry  $f : X \rightarrow Y$  is  $(\alpha, \gamma)$ -stable for some  $\alpha > 0$ ?*

However, Qian [30] presented a counterexample showing that if a separable Banach space  $Y$  contains an uncomplemented closed subspace  $X$  then for every  $\varepsilon > 0$  there is a standard  $\varepsilon$ -isometry  $f : X \rightarrow Y$  which is not stable. Cheng, Dong and Zhang [10] showed the following weak stability version.

**Theorem 1.3** (Cheng-Dong-Zhang). *Let  $X$  and  $Y$  be Banach spaces, and let  $f : X \rightarrow Y$  be a standard  $\varepsilon$ -isometry for some  $\varepsilon \geq 0$ . Then for every  $x^* \in X^*$ , there exists  $\phi \in Y^*$  with  $\|\phi\| = \|x^*\| \equiv r$  such that*

$$|\langle \phi, f(x) \rangle - \langle x^*, x \rangle| \leq 4\varepsilon r, \text{ for all } x \in X.$$

For study of the stability of  $\varepsilon$ -isometries of Banach spaces, the following question was proposed in [11].

**Problem 1.4.** *Is there a characterization for the class of Banach spaces  $\mathcal{X}$  satisfying given any  $X \in \mathcal{X}$  and Banach space  $Y$ , the pair  $(X, Y)$  is  $((\alpha, \gamma)$ -, resp.) stable?*

Every space  $X$  of this class is said to be a universally  $((\alpha, \gamma)$ -, resp.) left-stable space.

On one hand, Cheng, Dai, Dong et.al. [11] proved that every injective Banach space is a universally left-stable space. On the other hand, the first two authors Cheng and Dai, together with others [5] showed that every universally left-stable space is just a cardinality injective Banach space (i.e., a Banach space which is complemented in every superspace with the same cardinality) and they also showed that a dual space is injective if and only if it is a universally left-stable space, and further asked if every universally left-stable space is an injective Banach space. In Section 3, we will show that the second dual of a universally left-stable space is injective and that for a dual space, cardinality injectivity, separably injectivity and injectivity are equivalent to universal left-stability.

The following Problem 1.5 is also very natural.

**Problem 1.5.** *Is there a characterization for the class of Banach spaces  $\mathcal{S}$  satisfying given any  $X \in \mathcal{S}$  and separable Banach space  $Y$ , the pair  $(X, Y)$  is  $((\alpha, \gamma)$ -, resp.) stable?*

Every space  $X$  of this class is said to be a separably universally (resp.  $(\alpha, \gamma)$ ) left-stable space. In Section 4, we will show that all of these spaces of the class  $\mathcal{S}$  coincide with separably injective Banach spaces. We here refer

the reader to a very excellent paper [4] by Avilés-Sánchez-Castillo-González-Moreno for further information about injective Banach spaces and separably injective Banach spaces where they resolve a long standing problem proposed by Lindenstrauss in the middle sixties.

In this paper, we first consider a weaker version of Problem 1.2 in Section 2. That is Theorem 2.4, by which we discuss Qian's problem for  $\mathcal{L}_{\infty, \lambda}$ -spaces and  $C(K)$ -spaces, and then conclude all of the results in Section 3 and Section 4 stated as follows.

In section 2, we use Theorem 1.3 and Lemma 2.2 to prove Theorem 2.4 that if  $X, Y$  are Banach spaces, and  $Y^*$  has the point of  $w^*$ -norm continuity property (in short,  $w^*$ -PCP) or  $Y$  is separable, then there exists a  $w^*$ -dense  $G_\delta$  subset  $\Omega$  of  $\text{Ext}(B_{X^*})$  such that there is a bounded linear operator  $T : Y \rightarrow C(\Omega, \tau_{w^*})$  such that

$$\|Tf(x) - x\| \leq 4\epsilon, \text{ for all } x \in X.$$

In particular, we obtain a weak positive answer to Qian's problem for  $C(K)$ -spaces (see Corollary 2.5).

In section 3, combined Theorem 2.4 with some results from Johnson [22] and Avilés-Sánchez-Castillo-González-Moreno [4] we show that (a)  $X^{**}$  is an injective Banach space if  $X$  is universally left-stable. (b) If  $X^{**}$  is  $\lambda$ -injective, then for every standard  $\epsilon$ -isometry  $f : X \rightarrow Y$ , there is a bounded linear operator  $S : Y \rightarrow X^{**}$  with  $\|S\| \leq \lambda$  such that  $Sf - Id$  is uniformly bounded by  $4\lambda\epsilon$  on  $X$ . (c) If  $X$  is a  $\mathcal{L}_{\infty, \lambda}$ -space, then for every standard  $\epsilon$ -isometry  $f : X \rightarrow Y$ , there is a bounded linear operator  $T : Y \rightarrow X^{**}$  such that  $Tf - Id$  is uniformly bounded by  $4\lambda\epsilon$  on  $X$ . If, in addition,  $X$  is isomorphic to a dual space  $M^*$ , then  $X$  is universally  $(\lambda\alpha, 4\lambda\alpha)$  left-stable for each  $\alpha > d(X, M^*)$ , which further yields that  $X$  is  $\lambda\alpha$ -injective. Therefore, a dual space is separably injective if and only if it is universally left-stable.

In section 4, combined Theorem 2.4, together with some results from [24] by Johnson-Oikhberg (see also Rosenthal [29], Sánchez[31] and Castillo-Moreno [9]) and from [4] by Avilés-Sánchez-Castillo-González-Moreno, a quantitative characterization of separably injective Banach spaces is given. That is, we show that (i) if  $X$  is a  $\lambda$ -separably injective Banach space, then the pair  $(X, Y)$  is  $(3\lambda, 12\lambda)$  stable for every separable Banach space  $Y$ . (ii) If the pair  $(X, Y)$  is  $(\lambda, 4\lambda)$  stable for every separable Banach space  $Y$ , then  $X$  is a  $\lambda$ -separably injective Banach space. For example, (a) for every compact  $F$ -space  $K$  (resp. compact  $K$  of height  $n$ ), the pair  $(C(K), Y)$  is  $(3, 12)$

(resp.  $(6n - 3, 24n - 12)$ ) stable for every separable Banach space  $Y$ . In particular,  $\ell_\infty/c_0$  is separably universally  $(3, 12)$  left-stable. (b) If  $\{E_i\}_{i \in \Lambda}$  is a family of  $\lambda$ -separably injective space, then the pair  $((\sum_{i \in \Lambda} E_i)_{\ell_\infty}, Y)$  (resp.  $((\sum_{i \in \Lambda} E_i)_{c_0}, Y)$ ) is  $(3\lambda, 12\lambda)$  (resp.  $(6\lambda^2, 24\lambda^2)$ ) stable for every separable Banach space  $Y$ . (iii) In particular, by the Cheng-Dong-Zhang theorem (Theorem 1.3) we prove a sharpen quantitative and generalized Sobczyk theorem [33], that is, Theorem 4.6 if either  $E_i = c_0(\Gamma_i)$  or  $E_i$  is  $\lambda$ -injective for each  $i \in \Lambda$ .

All symbols and notations in this paper are standard. We use  $X$  to denote a real Banach space and  $X^*$  its dual.  $B_X$ ,  $\text{Ext}(B_{X^*})$  and  $S_X$  denote the closed unit ball of  $X$ , the set of all extremal points of  $B_{X^*}$  and the unit sphere of  $X$ , respectively. Given a bounded linear operator  $T : X \rightarrow Y$ ,  $T^* : Y^* \rightarrow X^*$  stands for its conjugate operator. For a subset  $A \subset X$ ,  $2^A$ ,  $\overline{A}$ ,  $\text{card}(A)$  and  $\text{dens}(A)$  stand respectively for the power set of  $A$ , the closure of  $A$ , the cardinality of  $A$ , the density character of  $A$ . We denote by  $d(X, Y) = \inf\{\|T\| \cdot \|T^{-1}\| : T \text{ is an isomorphism between } X \text{ and } Y\}$  the Banach-Mazur distance between  $X$  and  $Y$ .

## 2 $\varepsilon$ -Isometric embedding into Banach spaces whose dual has the $w^*$ -PCP

Recall that  $\mathcal{S}$  is the class of Banach spaces satisfying given any  $X \in \mathcal{S}$  and separable Banach space  $Y$ , the pair  $(X, Y)$  is  $((\alpha, \gamma)$ -, resp.) stable. Every space  $X$  of this class is said to be a separably universally  $((\alpha, \gamma)$ -, resp.) left-stable space. In this section, we will consider a weaker version of Problem 1.2. That is, Theorem 2.4, by which we will discuss Qian's problem for  $C(K)$ -spaces (Corollary 2.5) and  $\mathcal{L}_{\infty, \lambda}$ -spaces (Theorem 3.13), and then show that for a dual space, cardinality injectivity, separably injectivity and injectivity are equivalent to universal left-stability. Moreover, we completely solve Problem 1.5 in Section 4. That is, we prove that all of these spaces of the class  $\mathcal{S}$  coincide with separably injective Banach spaces.

Recall that a dual Banach space  $Y^*$  is said to have the point of weak star to norm continuity property (in short,  $w^*$ -PCP) if every nonempty bounded subset of  $Y^*$  admits relative weak star neighborhoods of arbitrarily small norm diameter. For example, if  $Y$  is an Asplund space, then  $Y^*$  has the  $w^*$ -PCP (see, for instance, [28]).

Recall that a set valued mapping  $F : X \rightarrow 2^Y$  is said to be usco provided it is nonempty compact valued and upper semicontinuous, i.e.,  $F(x)$

is nonempty compact for each  $x \in X$  and  $\{x \in X : F(x) \subset U\}$  is open in  $X$  whenever  $U$  is open in  $Y$ . We say that  $F$  is usco at  $x \in X$  if  $F$  is nonempty compact valued and upper semicontinuous at  $x$ , i.e., for every open set  $V$  of  $Y$  containing  $F(x)$  there exists a open neighborhood  $U$  of  $X$  such that  $F(U) \subset V$ . Therefore,  $F$  is usco if and only if  $F$  is usco at each  $x \in X$ .

Recall that a mapping  $\varphi : X \rightarrow Y$  is called a selection of  $F$  if  $\varphi(x) \in F(x)$  for each  $x \in X$ , moreover, we say  $\varphi$  is a continuous (linear) selection of  $F$  if  $\varphi$  is a continuous (linear) mapping. We denote the graph of  $F$  by  $G(F) \equiv \{(x, y) \in X \times Y : y \in F(x)\}$ , we write  $F_1 \subset F_2$  if  $G(F_1) \subset G(F_2)$ . A usco mapping  $F$  is said to be minimal if  $E = F$  whenever  $E$  is a usco mapping and  $E \subset F$  (see, for instance, [12], [28, page 19, 102-109]).

The following Problem 2.1 is equivalent to Problem 1.2.

**Problem 2.1.** *Does there exist a constant  $\gamma > 0$  depending only on  $X$  and  $Y$  with the following property: For each  $\varepsilon$ -isometry  $f : X \rightarrow Y$  with  $f(0) = 0$  there is a  $w^* - w^*$  continuous linear selection  $Q$  of the set-valued mapping  $\Phi$  from  $X^*$  into  $2^{L(f)^*}$  defined by*

$$\Phi(x^*) = \{\phi \in L(f)^* : |\langle \phi, f(x) \rangle - \langle x^*, x \rangle| \leq \gamma \|x^*\| \varepsilon, \text{ for all } x \in X\},$$

where  $L(f) = \overline{\text{span}} f(X)$ ?

The following Lemma 2.2 was motivated by Dai et.al. in [12, Lemma 4.2]. By an analogous argument we conclude the result on  $w^* - w^*$  usco mappings, which will be used to prove the main results.

**Lemma 2.2.** *Suppose that  $X, Y$  are Banach spaces. Let  $\varepsilon \geq 0$ . Assume that  $f$  is a  $\varepsilon$ -isometry from  $X$  into  $Y$  with  $f(0) = 0$ ,  $H$  is a Baire subspace contained in  $S_{X^*}$ . If we define a set-valued mapping  $\Phi_1 : S_{X^*} \rightarrow 2^{S_{L(f)^*}}$  by*

$$\Phi_1(x^*) = \{\phi \in S_{L(f)^*} : |\langle \phi, f(x) \rangle - \langle x^*, x \rangle| \leq 4\varepsilon, \text{ for all } x \in X\},$$

where  $L(f) = \overline{\text{span}} f(X)$ . Then

- (i)  $\Phi_1$  is convex  $w^*$ -usco at each point of  $S_{X^*}$ .
- (ii) There exists a minimal convex  $w^* - w^*$  usco mapping contained in  $\Phi_1$ .
- (iii) If, in addition,  $Y^*$  has the  $w^*$ -PCP (especially, if  $Y$  is an Asplund space) or  $Y$  is separable, then there exists a selection  $Q$  of  $\Phi_1$  such that  $Q$  is  $w^* - w^*$  continuous on a  $w^*$ -dense  $G_\delta$  subset of  $H$ .

*Proof.* (i) It follows easily from [12, Lemma 4.2 (i)].

(ii) By Zorn Lemma (see [12, Lemma 4.2 (ii)] or [28, Prop.7.3, p.103]) there exists a minimal convex  $w^* - w^*$  usco mapping contained in  $\Phi_1$ .

(iii) By (ii) there is a minimal convex  $w^* - w^*$  usco mapping  $F \subset \Phi_1$ , and  $H$  itself is a Baire space with respect to  $w^* -$  topology, and  $Y^*$  has the  $w^*$ -PCP (especially, if  $Y$  is an Asplund space) or  $Y$  is separable, which follows easily from [28, Lemma 7.14, p.106-107] and [12, Lemma 4.2 (iii)].  $\square$

**Remark 2.3.** The above Lemma 2.2 also holds if we substitute  $Y^*$  and  $S_{Y^*}$  for  $L(f)^*$  and  $S_{L(f)^*}$ , respectively.

**Theorem 2.4.** *Suppose that  $X, Y$  are Banach spaces. Let  $\varepsilon \geq 0$ . Assume that  $f$  is a  $\varepsilon$ -isometry from  $X$  into  $Y$  with  $f(0) = 0$ . Then*

(1) *for every  $w^*$ -dense subset  $\Omega \subset \text{Ext}(B_{X^*})$  there is a bounded linear operator  $T : Y \rightarrow \ell_\infty(\Omega)$  such that*

$$\|Tf(x) - x\| \leq 4\varepsilon, \text{ for all } x \in X.$$

(2) *If  $Y^*$  has the  $w^*$ -PCP or  $Y$  is separable, then there exists a  $w^*$ -dense  $G_\delta$  subset  $\Omega \subset \text{Ext } B_{X^*}$  such that there is a bounded linear operator  $T : Y \rightarrow C(\Omega, \tau_{w^*})$  such that*

$$\|Tf(x) - x\| \leq 4\varepsilon, \text{ for all } x \in X.$$

*Proof.* (1) By Theorem 1.3, for every  $x^* \in \Omega$ , there exists a functional  $Q(x^*) \in S_{Y^*}$  such that

$$|\langle Q(x^*), f(x) \rangle - \langle x^*, x \rangle| \leq 4\varepsilon, \text{ for all } x \in X.$$

We now define a mapping  $T : Y \rightarrow \ell_\infty(\Omega)$  by

$$T(y) = \{Q(x^*)(y)\}_{x^* \in \Omega}.$$

It is clear that  $T$  is a bounded linear operator with norm one and

$$\|Tf(x) - x\| = \sup_{x^* \in \Omega} |Q(x^*)f(x) - x^*(x)| \leq 4\varepsilon, \text{ for all } x \in X.$$

(2) Since  $\text{Ext}(B_{X^*})$  itself is a Baire space in its relative  $w^*$ -topology (see [20, p.217, line 17-19]), it follows from Lemma 2.2 that there is a  $w^*$ -dense  $G_\delta$  subset  $\Omega$  in  $\text{Ext}(B_{X^*})$  such that there is a  $w^* - w^*$  continuous selection  $Q$  of  $\Phi_1$  on  $\Omega$  satisfying that for every  $x \in X$  and  $x^* \in \Omega$ , the following inequality holds :

$$|\langle Q(x^*), f(x) \rangle - \langle x^*, x \rangle| \leq 4\varepsilon.$$

Let  $T : Y \rightarrow \ell_\infty(\Omega)$  be defined as in (i). Therefore,  $T(y) \in C(\Omega, \tau_{w^*})$  and

$$\|Tf(x) - x\| \leq 4\varepsilon, \text{ for all } x \in X.$$

□

**Corollary 2.5.** *Suppose that  $X = C(K)$  for a compact Hausdorff space  $K$  and  $Y^*$  has the  $w^*$ -PCP (especially, if  $Y$  is an Asplund space) or  $Y$  is separable. Let  $\varepsilon \geq 0$ . Assume that  $f$  is a standard  $\varepsilon$ -isometry from  $X$  into  $Y$ . Then there exists a dense  $G_\delta$  subset  $\Omega$  of  $K$  such that there is a bounded linear operator  $T : Y \rightarrow C(\Omega)$  such that  $Tf - Id$  is uniformly bounded by  $4\varepsilon$  on  $X$ .*

*Proof.* It suffices to note that  $\text{Ext}(B_{X^*}) = \{\pm\delta_t : t \in K\}$  and  $\{\delta_t : t \in K\}$  is a compact Baire space norming for  $X$ , and then apply Lemma 2.2 and Theorem 2.4 to conclude the results we desired by substituting  $\{\delta_t : t \in K\}$  respectively for  $H$  and  $\text{Ext}(B_{X^*})$  everywhere.

□

### 3 A quantitative characterization of separably injective dual spaces

This section is based on a communication with W.B. Johnson, and the author would like to thank him for discussion. In this section, combined Theorem 2.4 with some results from Avilés-Sánchez-Castillo-González-Moreno [4] and Johnson [22] we show that

- (a)  $X^{**}$  is an injective Banach space if  $X$  is universally left-stable.
- (b) If  $X^{**}$  is  $\lambda$ -injective, then for every standard  $\varepsilon$ -isometry  $f : X \rightarrow Y$ , there is a bounded linear operator  $S : Y \rightarrow X^{**}$  with  $\|S\| \leq \lambda$  such that  $Sf - Id$  is uniformly bounded by  $4\lambda\varepsilon$  on  $X$ .
- (c) If  $X$  is a  $\mathcal{L}_{\infty, \lambda}$ -space, then for every standard  $\varepsilon$ -isometry  $f : X \rightarrow Y$ , there is a bounded linear operator  $S : Y \rightarrow X^{**}$  with  $\|S\| \leq \lambda$  such that  $Sf - Id$  is uniformly bounded by  $4\lambda\varepsilon$  on  $X$ . If, in addition,  $X$  is isomorphic to a dual space  $M^*$ , then  $X$  is universally  $(\lambda\alpha, 4\lambda\alpha)$  left-stable for each  $\alpha > d(X, M^*)$ , which further yields that it is  $\lambda\alpha$ -injective. Therefore, a dual space is separably injective if and only if it is universally left-stable.

Recall that a Banach space  $X$  is said to be  $\lambda$ -(resp. separably injective) injective if it has the following extension property: Every bounded linear operator  $T$  from a closed subspace of a (resp. separable) Banach space into  $X$  can be extended to be a bounded operator on the whole space with its



norm at most  $\lambda\|T\|$  (see, for instance, [1], [4], [15], [35], [36]). In this case,  $X$  is said to be injective (resp. separably injective) if it is  $\lambda$ -(resp. separably injective) injective for some  $\lambda \geq 1$ .

The following Proposition 3.1 follows easily from Remark 3.3.

**Proposition 3.1.** *A (resp. separable) Banach space  $X$  is  $\lambda$ -(resp. separably injective) injective if and only if it is  $\lambda$ -complemented in every (resp. separable) superspace (i.e., a normed linear space which contains  $X$ ).*

The following Proposition 3.2 was proved by Avilés, Sánchez, Castillo, González and Moreno (see [4, Prop. 3.2]).

**Proposition 3.2.** (1) If a Banach space  $X$  is  $\lambda$ -separably injective, then it is  $3\lambda$ -complemented in every superspace  $Y$  such that  $Y/X$  is separable.

(2) If a Banach space  $X$  is  $\lambda$ -complemented in every superspace  $Y$  such that  $Y/X$  is separable, then  $X$  is  $\lambda$ -separably injective.

**Remark 3.3.** For any set  $\Gamma$ , that  $\ell_\infty(\Gamma)$  is 1-injective follows from the Hahn-Banach theorem.

**Definition 3.4.** A Banach space  $X$  is said to be cardinality injective if it is complemented in every superspace (a normed linear space containing  $X$ ) with the same cardinality.

**Proposition 3.5.** *A Banach space  $X$  is cardinality injective if and only if every bounded linear operator  $T$  from a subspace  $Z$  of a normed linear space  $Y$  with  $\text{card}(Y) \leq \text{card}(X)$  into  $X$  can be extended to be a bounded operator on the whole space with its norm at most  $\lambda\|T\|$ , where  $\lambda$  depends only on  $X$ . In this case, we say  $X$  is  $\lambda$ -cardinality injective.*

*Proof.* Sufficiency. It is trivial.

Necessity. It is clear that  $J(X)$  is also cardinality injective where  $J$  is the canonical embedding from  $X$  into  $\ell_\infty(B_{X^*})$ . Let  $\tilde{S} : Y \rightarrow \ell_\infty(B_{X^*})$  be a norm-preserving extension of operator  $J \cdot T : Z \rightarrow \ell_\infty(B_{X^*})$ . Let  $Y' = \text{span} \{J(X) \cup \tilde{S}(Y)\}$ . So there is a projection from  $Y'$  onto  $J(X)$ . Hence  $\tilde{T} = J^{-1} \cdot P \cdot \tilde{S}$  is an extension of  $T$  such that  $\|\tilde{T}\| \leq \|P\|\|T\|$ .

We now show that there is a constant  $\lambda$  depending only on  $X$  such that for every  $Y$  with  $\text{card } Y \leq \text{card } X$ , every subspace  $Z$  and every operator  $T : Z \rightarrow X$ , there is an extension  $\tilde{T}$  of  $T$  satisfying  $\|\tilde{T}\| \leq \lambda\|T\|$ . To the contrary, for each  $n \in \mathbb{N}$  there exist a normed linear space  $Y_n$  with  $\text{card } Y_n \leq \text{card } X$ , a subspace  $Z_n$  of  $Y_n$  and an operator  $T_n : Z_n \rightarrow X$  such that for every extension  $\tilde{T}_n$  of  $T_n$ ,  $\|\tilde{T}_n\| \geq n\|T_n\|$ . Let  $Y = (\Sigma Y_n)_{c_{00}}$  endowed

the norm  $\|\cdot\|_{\ell_1}$  and  $Z = (\Sigma Z_n)_{c_{00}} \subset Y$ . Obviously,  $\text{card}(Y) \leq \text{card}(X)$  and let  $T : Z \rightarrow X$  be defined for all  $z = \{z_n\} \in Z$  by  $T(z) = \sum \frac{T_n z_n}{\|T_n\|}$  and  $\|T\| = 1$ . If  $\tilde{T}$  is an extension of  $T$ , then  $\|\tilde{T}\| \geq n$  for every  $n \in \mathbb{N}$ , which is a contradiction.

□

The following Lemma 3.6 follows from Qian's counterexample in [30] (see also [11]).

**Lemma 3.6.** *Let  $X$  be a closed subspace of Banach space  $Y$ . If  $\text{card}(X) = \text{card}(Y)$ , then for every  $\varepsilon > 0$  there is a standard  $\varepsilon$ -isometry  $f : X \rightarrow Y$  such that*

- (1)  $L(f) \equiv \overline{\text{span}} f(X) = Y$ ;
- (2)  $X$  is complemented whenever  $f$  is stable.

The following Lemma 3.7 and Lemma 3.8 are due to W.B.Johnson based on a communication (see [22]).

**Lemma 3.7.**  $\text{card}(X) = \text{dens}(X)^{\aleph_0}$ .

*Proof.* It is clear that  $\text{card}(X) \leq \text{dens}(X)^{\aleph_0}$ . It suffices to show that  $\text{card}(X) \geq \text{dens}(X)^{\aleph_0}$ . By the Riesz's lemma and axiom of choice, there exists a set  $\{x_i : 0 \preceq i \prec \text{dens}(X)\}$  such that for each  $1 \preceq j \prec \text{dens}(X)$ ,  $d(x_j, \text{span}\{x_i : i \prec j\}) > \frac{1}{2}$ . It follows for each  $i \neq j$  that  $\|x_i - x_j\| > \frac{1}{2}$ . We now define a mapping  $g$  for each  $i \in \mathbb{N}$  and  $0 \preceq n_i \prec \text{dens}(X)$  by  $g(\{x_{n_i}\}_{i=0}^\infty) = \sum_{i=0}^\infty \frac{1}{2^i} x_{n_i}$ . For each  $\{x_{n_i}\}_{i=0}^\infty \neq \{x_{m_i}\}_{i=0}^\infty$ , let  $k \in \mathbb{N}$  be the least cardinal number such that  $x_{n_i} \neq x_{m_i}$ . It follows from the triangle inequality that  $\|2^k \sum_{i=k}^\infty \frac{1}{2^{i-k}} x_{n_i} - 2^k \sum_{i=k}^\infty \frac{1}{2^{i-k}} x_{m_i}\| > 0$ . Hence  $\|g(\{x_{n_i}\}_{i=0}^\infty) - g(\{x_{m_i}\}_{i=0}^\infty)\| > 0$  and we complete the proof.

□

**Lemma 3.8.** *Every Banach space is linearly isometric to a subspace of some  $C(K)$ -space with the same cardinality, where  $K$  is a compact Hausdorff space.*

*Proof.* Let  $X$  be identified with a subspace of  $C(B_{X^*}, \tau_{w^*})$  denoted by  $J(X)$ :  $J(x)(x^*) = x^*(x)$  for all  $x^* \in B_{X^*}$ . Let  $X_0$  be a dense set of  $X$  such that  $\text{card}(X_0) = \text{dens}(X)$  by the well-ordering principle of cardinals. Let  $P(X_0)$  be defined to be a subspace consisting of all polynomials with rational coefficients by

$$P(X_0) \equiv \{q_m x_1^{p_1} x_2^{p_2} \cdots x_m^{p_m} : m, p_m \in \mathbb{N}, q_m \in \mathbb{Q} \text{ and } x_i \in J(X_0)\}.$$

By the Stone-Weierstrass theorem, the closure of  $P(X_0)$  forms a subalgebra which contains all constants and separates all points of  $B_{X^*}$ , hence  $\overline{P(X_0)} = C(B_{X^*}, \tau_{w^*})$ . It is easy to see that  $\text{card}(P(X_0)) = \text{card}(X_0)$ , thus  $\text{dens}(C(B_{X^*}, \tau_{w^*})) \leq \text{dens}(X)$ . Therefore, by Lemma 3.7,  $\text{card}(X) = \text{card}(C(K))$ , where  $K = B_{X^*}$  endowed the usual weak star topology  $\tau_{w^*}$ .

□

**Proposition 3.9.**  *$X$  is complemented in every complete superspace  $Y$  with the same cardinality if and only if it is complemented in every superspace which is isomorphic to a  $C(K)$  space with the same cardinality, where  $K$  is a compact Hausdorff space.*

*Proof.* It suffices to note that  $X \subset Y \subset C(B_{Y^*}, \tau_{w^*})$ .

□

**Lemma 3.10.** *Suppose that  $X$  is  $\lambda$ -cardinality injective. Then  $X^{**}$  is  $\lambda$ -injective. If, in addition,  $X$  is isomorphic to a dual space, then  $X$  is even an  $\alpha$ -injective Banach space for every  $\alpha > d(X, M^*)\lambda$ .*

*Proof.* By Lemma 3.8,  $X$  is  $\lambda$ -complemented in some  $C(K)$ -space for a compact Hausdorff space  $K$ . Hence  $X^{**}$  is  $\lambda$ -complemented in the 1-injective Banach space  $C(K)^{**}$ . Thus  $X^{**}$  is  $\lambda$ -injective Banach space. If, in addition,  $X$  is isomorphic to a dual space  $M^*$ , then  $X$  is even an  $\alpha$ -injective Banach space for every  $\alpha > d(X, M^*)\lambda$  since a dual space is complemented in its second dual.

□

**Theorem 3.11.** *Suppose that  $X$  is a Banach space such that for every Banach space  $Y$  and every standard  $\varepsilon$ -isometry  $f : X \rightarrow Y$ , there exist  $\gamma > 0$  and a bounded linear operator  $T : L(f) \rightarrow X$  satisfying that*

$$\|Tf(x) - x\| \leq \gamma\varepsilon, \quad \text{for all } x \in X.$$

*Then  $X^{**}$  is an injective Banach space. If, in addition,  $X$  is isomorphic to a dual space, then  $X$  is injective.*

*Proof.* Let  $Y = C(B_{X^*}, \tau_{w^*})$ . By Lemma 3.8,  $\text{card}(X) = \text{card}(Y)$ . Let  $f : X \rightarrow Y$  be defined as in Lemma 3.6. Thus,  $Y = L(f) \equiv \overline{\text{span}} f(X)$  and  $X$  is complemented in  $Y$ , hence that follows from Lemma 3.10.

□

By an analogous argument of Theorem 2.4 we have the following Corollary.

**Corollary 3.12.** *Suppose that  $X^{**}$  is  $\lambda$ -injective,  $Y$  is a Banach space, Let  $\varepsilon \geq 0$ . Assume that  $f$  is a standard  $\varepsilon$ -isometry from  $X$  into  $Y$ . Then there is a bounded linear operator  $S : Y \rightarrow X^{**}$  with  $\|S\| \leq \lambda$  such that*

$$\|Sf(x) - x\| \leq 4\lambda\varepsilon, \text{ for all } x \in X.$$

*Proof.* It suffices to note that  $X^{**}$  is  $\lambda$ -complemented in  $\ell_\infty(\Omega)$  for every norm-dense set of  $\text{Ext}(B_{X^*})$ . By an analogous argument of Theorem 2.4, there is a bounded linear operator  $T : Y \rightarrow \ell_\infty(\Omega)$  such that  $Tf - Id$  is uniformly bounded by  $4\varepsilon$  on  $X$ . Let  $S = PT : Y \rightarrow X^{**}$  for a projection  $P : \ell_\infty(\Omega) \rightarrow X^{**}$  with  $\|P\| \leq \lambda$ . Therefore,  $Sf - Id$  is uniformly bounded by  $4\lambda\varepsilon$ . □

Recall that a Banach space  $X$  is said to be a  $\mathcal{L}_{\infty,\lambda}$ -space if every finite dimensional subspace  $F$  of  $X$  is contained in another finite dimensional subspace  $E$  of  $X$  such that  $d(E, \ell_\infty^{\dim E}) \leq \lambda$  (see, for instance, [3], [4], [8]).

**Theorem 3.13.** *Suppose that  $X$  is a  $\mathcal{L}_{\infty,\lambda}$ -space and  $Y$  is a Banach space. Then*

(i) *for every standard  $\varepsilon$ -isometry  $f : X \rightarrow Y$ , there is a bounded linear operator  $T : Y \rightarrow X^{**}$  such that  $Tf - Id$  is uniformly bounded by  $4\lambda\varepsilon$  on  $X$ .*

(ii) *If, in addition,  $X$  is isomorphic to a dual space  $M^*$ , then  $X$  is universally  $(\lambda\alpha, 4\lambda\alpha)$  left-stable for each  $\alpha > d(X, M^*)$ . Hence,  $X$  is  $\lambda\alpha$ -injective.*

*Proof.* (i) By Theorem 2.4 (i), for every  $w^*$ -dense subset  $\Omega \subset \text{Ext}(B_{X^*})$  there is a bounded linear operator  $T : Y \rightarrow \ell_\infty(\Omega)$  such that  $Tf - Id$  is uniformly bounded by  $4\varepsilon$  on  $X$ . Let  $X = \cup_{i \in I} E_i$  be such that for every  $i, j \in (I, \succeq)$ ,  $i \succeq j$  if and only if  $E_i \supseteq E_j$  satisfying that for each  $i \in I$ ,  $\dim E_i < \infty$  and  $d(E_i, \ell_\infty^{\dim E_i}) \leq \lambda$ . Hence for each  $i \in I$ , there exists a projection  $P_i : \ell_\infty(\Omega) \rightarrow E_i$  such that  $\|P_i\| < \lambda + \frac{1}{1 + \dim E_i}$ . Since  $\{P_i\}_{i \in I}$  is uniformly bounded on  $B_{\ell_\infty(\Omega)^*}$ , it follows from the Arzelà-Ascoli theorem that there is a subnet  $\{\delta_i\}_{i \in \Lambda}$  of  $I$  for an partial order set  $\Lambda$  such that  $P : \ell_\infty(\Omega) \rightarrow X^{**}$  is well defined by

$$P(y) = w^* - \lim_{i \in \Lambda} P_{\delta_i}(y), \text{ for all } y \in \ell_\infty(\Omega),$$

which yields that

$$\|P\| \leq \lambda \text{ and } P|_X = Id.$$

Hence

$$\|PTf(x) - x\| \leq 4\varepsilon\lambda, \text{ for all } x \in X,$$

where  $PT : Y \rightarrow X$  with  $\|PT\| \leq \lambda$ .

(ii) By the assumption, there exists an isomorphism  $S : X \rightarrow M^*$  such that  $\|S\| \cdot \|S^{-1}\| < \alpha$ . Clearly,  $SP_i : \ell_\infty(\Omega) \rightarrow M^*$  is uniformly bounded on  $B_{\ell_\infty(\Omega)^*}$ . It follows from (i) that there is a subnet  $\{\delta_i\}_{i \in \Lambda}$  such that  $Q : \ell_\infty(\Omega) \rightarrow M^*$  is well defined by

$$Q(y) = w^* - \lim_{i \in \Lambda} SP_{\delta_i}(y), \text{ for all } y \in \ell_\infty(\Omega).$$

Hence  $S^{-1}QT : Y \rightarrow X$  is a bounded linear operator with  $\|S^{-1}QT\| \leq \alpha\lambda$  such that  $S^{-1}Q|_X = Id$  and

$$\|S^{-1}QTf(x) - x\| \leq 4\varepsilon\alpha\lambda, \text{ for all } x \in X.$$

Thus, it follows from Lemma 3.10 that  $X$  is  $\lambda\alpha$  injective and we complete the proof.  $\square$

Combined Theorem 3.13 with Theorem 3.11, we have the following Corollary 3.14.

**Corollary 3.14.** *A dual Banach space is separably injective if and only if it is universally left-stable.*

*Proof.* It suffices to note that a dual space is complemented in its second dual, hence sufficiency follows from Theorem 3.11. Note that a  $\lambda$ -separably injective Banach space is  $\mathcal{L}_{\infty,9\lambda^+}$ -space (see [4, p.199, Prop.3.5 (a)]). Hence, necessity follows from Theorem 3.13 (ii).  $\square$

**Remark 3.15.** For a dual space, cardinality injectivity, separably injectivity and injectivity are equivalent to universal left-stability.

## 4 A quantitative characterization of separably injective Banach spaces

In this section, combined Theorem 2.4 (ii) with some results from [24] by Johnson-Oikhberg (Lindenstrass[25], Rosenthal [29], Sánchez [31] and Castillo-Moreno [9]) and from [4] by Avilés-Sánchez-Castillo-González-Moreno, we conclude a quantitative characterization of separably injective Banach space which completely solves Problem 1.5. That is, we show that:

(i) if  $X$  is a  $\lambda$ -separably injective Banach space, then the pair  $(X, Y)$  is  $(3\lambda, 12\lambda)$  stable for every separable Banach space  $Y$ ;

(ii) If the pair  $(X, Y)$  is  $(\lambda, 4\lambda)$  stable for every separable Banach space  $Y$ , then  $X$  is a  $\lambda$ -separably injective Banach space;

As a corollary, (a) for every compact  $F$ -space  $K$  (for example,  $K = \beta\mathbb{N} \setminus \mathbb{N}$ ), the pair  $(C(K), Y)$  (resp.  $(\ell_\infty/c_0, Y)$ ) is  $(3, 12)$  stable for every separable Banach space  $Y$ ;

(b) For every compact space  $K$  of height  $n$ , the pair  $(C(K), Y)$  is  $(6n - 3, 24n - 12)$  stable for every separable Banach space  $Y$ ;

(c) If  $\{E_i\}_{i \in \Lambda}$  is a family of  $\lambda$ -separably injective space, then the pair  $((\sum_{i \in \Lambda} E_i)_{\ell_\infty}, Y)$  (resp.  $((\sum_{i \in \Lambda} E_i)_{c_0}, Y)$ ) is  $(3\lambda, 12\lambda)$  (resp.  $(6\lambda^2, 24\lambda^2)$ ) stable for every separable Banach space  $Y$ ;

(iii) If either  $E_i = c_0(\Gamma_i)$  or  $E_i$  is a  $\lambda$ -injective Banach space for each  $i \in \Lambda$ , then by Theorem 1.3 we have a sharpen estimate for the constant pair  $(\alpha, \gamma)$  in Theorem 4.6, which could be seen as a quantitative and generalized Sobczyk theorem.

**Theorem 4.1.** (i) If  $X$  is a  $\lambda$ -separably injective Banach space, then the pair  $(X, Y)$  is  $(3\lambda, 12\lambda)$  stable for every separable Banach space  $Y$ .

(ii) If the pair  $(X, Y)$  is  $(\lambda, 4\lambda)$  stable for every separable Banach space  $Y$ , then  $X$  is a  $\lambda$ -separably injective Banach space.

*Proof.* (i) Since  $Y$  is separable, it follows from Theorem 2.4 (ii) that for every  $w^*$ -dense subset  $\Omega \subset \text{Ext}(B_{X^*})$ , there is a bounded linear operator  $T : Y \rightarrow C(\Omega, \tau_{w^*})$  such that

$$\|Tf(x) - x\| \leq 4\varepsilon, \text{ for all } x \in X.$$

Hence, it could be reduced to ask if  $X$  is complemented in  $Z = \overline{\text{span}} \{Tf(X) \cup X\}$ . It follows from the continuity of  $T$  that  $Z/X$  is separable quotient space since  $Y$  is separable. Since  $X$  is  $\lambda$ -separably injective, it follows from Proposition 3.2 that  $X$  is  $3\lambda$ -complemented in  $Z$ . Therefore, there is a bounded linear operator  $P : Z \rightarrow X$  with  $\|P\| \leq 3\lambda$  such that

$$\|PTf(x) - x\| = \|PTf(x) - Px\| \leq 12\varepsilon, \text{ for all } x \in X,$$

where  $PT : L(f) \rightarrow X$  satisfies that  $\|PT\| \leq 3\lambda$ .

(ii) By Proposition 3.2, it suffices to show that  $X$  is  $\lambda$ -complemented in every superspace  $Y$  such that  $Y/X$  is separable. Let  $Y = X + Y/X$  be the algebraic direct sum. Since  $Y/X$  is separable,  $\text{card}(X) = \text{card}(Y)$ . It follows from Qian's counterexample (i.e., Lemma 3.6) that there is an

isometry  $f : X \rightarrow Y$  such that  $Y = L(f)$  and  $f(0) = 0$ . Hence by the assumption, there is a projection  $P : Y \rightarrow X$  with  $\|P\| \leq \lambda$  such that

$$\|Pf(x) - x\| \leq 4\varepsilon, \text{ for all } x \in X,$$

and we complete the proof.  $\square$

Recall that a compact Hausdorff space  $K$  is said to be an  $F$ -space if disjoint open  $F_\sigma$  sets have disjoint closures. For example,  $\beta\mathbb{N}$ , the Čech-Stone compactification of  $\mathbb{N}$  and  $\beta\mathbb{N} \setminus \mathbb{N}$  are  $F$ -spaces. Since  $C(K)$  is 1-separably injective for every  $F$ -space  $K$  (see, for instance, [4, p.202-203], [25]), we have

**Corollary 4.2.** *For every compact  $F$ -space  $K$  (for example,  $K = \beta\mathbb{N} \setminus \mathbb{N}$ ), the pair  $(C(K), Y)$  (resp.  $(\ell_\infty/c_0, Y)$ ) is  $(3, 12)$  stable for every separable Banach space  $Y$ .*

*Proof.* It is sufficient to note that  $\ell_\infty/c_0$  is linearly isometric to  $C(\beta\mathbb{N} \setminus \mathbb{N})$ .  $\square$

Recall that a compact space  $K$  has height  $n$  if  $K^{(n)} = \emptyset$ , where we write  $K'$  for the derived set of  $K$  and  $K^{(n+1)} = (K^{(n)})'$ . Since  $C(K)$  is  $(2n - 1)$ -separably injective for every  $K$  of height  $n$  (see, for instance, [4, p.203]), we have

**Corollary 4.3.** *For every compact space  $K$  of height  $n$ , the pair  $(C(K), Y)$  is  $(6n - 3, 24n - 12)$  stable for every separable Banach space  $Y$ .*

Combined Theorem 4.1 with the results of Johnson-Oikhberg [24] that for every family of  $\lambda$ -separably injective spaces  $\{E_i\}_{i \in \Lambda}$ ,  $(\sum_{i \in \Lambda} E_i)_{\ell_\infty}$  and  $(\sum_{i \in \Lambda} E_i)_{c_0}$  are respectively  $\lambda$ -separably injective and  $2\lambda^2$ -separably injective, which was also proved by Rosenthal [29], Sánchez [31] and Castillo-Moreno [9] with the estimates for the constant, respectively  $\lambda(1+\lambda)^+$ ,  $(3\lambda^2)^+$  and  $6(1+\lambda)$ , we have the following corollaries.

**Corollary 4.4.** *The pair  $((\sum_{i \in \Lambda} E_i)_{\ell_\infty}, Y)$  is  $(3\lambda, 12\lambda)$  stable for every separable Banach space  $Y$ , where  $\{E_i\}_{i \in \Lambda}$  is a family of  $\lambda$ -separably injective spaces.*

**Corollary 4.5.** *The pair  $((\sum_{i \in \Lambda} E_i)_{c_0}, Y)$  is  $(6\lambda^2, 24\lambda^2)$  (resp.  $(3\lambda(1+\lambda)^+, 12\lambda(1+\lambda)^+)$ ,  $((9\lambda^2)^+, (36\lambda^2)^+)$  and  $(18(1+\lambda), 72(1+\lambda))$ ) stable for every separable Banach space  $Y$ , where  $\{E_i\}_{i \in \Lambda}$  is a family of  $\lambda$ -separably injective spaces.*

If either  $E_i = c_0(\Gamma_i)$  or  $E_i$  is a  $\lambda$ -injective Banach spaces for each  $i \in \Lambda$ , then by Theorem 1.3 we have the following Theorem 4.6 which gives a sharpen estimate for the constant pair  $(\alpha, \gamma)$  by contrast with Corollary 4.4 and Corollary 4.5, respectively. In some sense, it could be seen as a quantitative and generalized Sobczyk theorem [33].

**Theorem 4.6.** *Let  $\Lambda$  and  $\Gamma_i$  for each  $i \in \Lambda$  are index sets. Suppose that one of the following three statements holds*

- i)  $X$  is isomorphic to  $Z = (\sum_{i \in \Lambda} c_0(\Gamma_i))_{\ell_\infty}$  and  $\lambda > d(X, Z)$ ;
- ii)  $X$  is isomorphic to  $Z = (\sum_{i \in \Lambda} \ell_\infty(\Gamma_i))_{c_0}$  and  $\lambda > d(X, Z)$ ;
- iii)  $X = (\sum_{i \in \Lambda} E_i)_{c_0}$  and  $\{E_i\}_{i \in \Lambda}$  is a family of  $\lambda$ -injective Banach spaces.

*Then  $(X, Y)$  is  $(2\lambda, 8\lambda)$ -stable for every separable Banach space  $Y$ .*

*Proof.* i) Let  $X$  be a Banach space isomorphic to  $(\sum_{i \in \Lambda} c_0(\Gamma_i))_{\ell_\infty}$  and  $T : X \rightarrow (\sum_{i \in \Lambda} c_0(\Gamma_i))_{\ell_\infty}$  be an isomorphism such that  $\|T\| \cdot \|T^{-1}\| < \lambda$ . For each  $n \in \Lambda$  and  $m \in \Gamma_n$ , let  $e_{nm} \in (\sum_{i \in \Lambda} c_0(\Gamma_i))_{\ell_\infty}$  with the standard biorthogonal functionals  $e_{nm}^* \in (\sum_{i \in \Lambda} c_0(\Gamma_i))_{\ell_\infty}^*$  such that  $e_{ij}^*(e_{nm}) = \delta_{in}\delta_{jm}$ . For all  $n \in \Lambda$  and  $m \in \Gamma_n$ , let  $x_{nm} \in X$  be such that  $T(x_{nm}) = e_{nm}$ . Let  $T^* : Z^* \rightarrow X^*$  be the conjugate operator of  $T$ . Then

$$T(x) = \left\{ \sum_{m \in \Gamma_n} (T^* e_{nm}^*)(x) e_{nm} \right\}_{n \in \Lambda}$$

and

$$x = T^{-1} \left\{ \sum_{m \in \Gamma_n} (T^* e_{nm}^*)(x) e_{nm} \right\}_{n \in \Lambda}, \text{ for all } x \in X.$$

For all  $n \in \Lambda$  and  $m \in \Gamma_n$ , let  $x_{nm}^* = T^* e_{nm}^* \in \|T\| B_{X^*}$ . It follows from Theorem 1.3 that for every  $n \in \Lambda$  and  $m \in \Gamma_n$ , there exists a functional  $\phi_{nm} \in \|T\| B_{Y^*}$  with  $\|\phi_{nm}\| = \|x_{nm}^*\|$  such that

$$(4.1) \quad |\langle \phi_{nm}, f(x) \rangle - \langle x_{nm}^*, x \rangle| \leq 4\epsilon \|T\|, \text{ for all } x \in X.$$

It follows from the  $w^* - w^*$  continuity of  $T^*$  that for each  $n \in \Lambda$ ,  $x_{nm}^* \rightarrow 0$  in the  $w^*$ -topology of  $X^*$ . Since  $e_{nm}^* \rightarrow 0$  in the  $w^*$ -topology of  $Z^*$ . Let

$$K = \{\psi \in \|T\| B(Y^*) : |\langle \psi, f(x) \rangle| \leq 4\epsilon \|T\|, \text{ for all } x \in X\}.$$

Then  $K$  is a nonempty  $w^*$ -compact subset of  $Y^*$ . Since  $Y$  is separable,  $(\|T\| B_{Y^*}, w^*)$  is metrizable. Let  $d$  be a metric such that  $(\|T\| B_{Y^*}, d)$  is homeomorphic to  $(\|T\| B_{Y^*}, w^*)$ . Since for each  $n \in \Lambda$ ,  $(x_{nm}^*)$  is a  $w^*$ -null net in  $X^*$ , inequality (4.1) implies that for each  $n \in \Lambda$ , every  $w^*$ -cluster point



$\phi$  of  $(\phi_{nm})$  is in  $K$  such that  $\|\phi\| \leq \|T\|$ , which yields that  $d(\phi_{nm}, K) \rightarrow 0$  for each  $n \in \Lambda$ . Hence, for each  $n \in \Lambda$ , there is a net  $(\psi_{nm}) \subset K$  such that  $d(\phi_{nm}, \psi_{nm}) \rightarrow 0$ , or equivalently,  $\phi_{nm} - \psi_{nm} \rightarrow 0$  in the  $w^*$ -topology of  $Y^*$ . Let  $S : Y \rightarrow X$  be defined for every  $y \in Y$  by

$$S(y) = T^{-1} \left\{ \sum_{m \in \Gamma_n} \langle \phi_{nm} - \psi_{nm}, y \rangle e_{nm} \right\}_{n \in \Lambda} \in X.$$

Hence

$$\|S\| \leq 2\|T\| \cdot \|T^{-1}\| < 2\lambda$$

and

$$\begin{aligned} \|Sf(x) - x\| &= \|T^{-1} \left\{ \sum_{m \in \Gamma_n} \langle \phi_{nm} - \psi_{nm}, f(x) \rangle e_{nm} \right\}_{n \in \Lambda} - T^{-1} \left\{ \sum_{m \in \Gamma_n} \langle x_{nm}^*, x \rangle e_{nm} \right\}_{n \in \Lambda}\| \\ &\leq \|T^{-1}\| \sup_{n \in \Lambda} \left( \left\| \sum_{m \in \Gamma_n} \langle \phi_{nm} - \psi_{nm}, f(x) \rangle e_{nm} - \sum_{m \in \Gamma_n} \langle x_{nm}^*, x \rangle e_{nm} \right\| \right) \\ &\leq \|T^{-1}\| \cdot \sup_{n \in \Lambda} \sup_{m \in \Gamma_n} |\langle \phi_{nm}, f(x) \rangle - \langle x_{nm}^*, x \rangle - \langle \psi_{nm}, f(x) \rangle| \\ &\leq \|T^{-1}\| \left( \sup_{n \in \Lambda} \sup_{m \in \Gamma_n} |\langle \phi_{nm}, f(x) \rangle - \langle x_{nm}^*, x \rangle| + \sup_{n \in \Lambda} \sup_{m \in \Gamma_n} |\langle \psi_{nm}, f(x) \rangle| \right) \\ &\leq 8\varepsilon \|T\| \cdot \|T^{-1}\| < 8\varepsilon \lambda. \end{aligned}$$

ii-iii) For each  $i \in \Lambda$ ,  $\Gamma_i$  denotes by  $B_{E_i^*}$ . It suffices to show this case that  $X = (\sum_{i \in \Lambda} E_i)_{c_0}$ . Let  $J : X = (\sum_{i \in \Lambda} E_i)_{c_0} \rightarrow (\sum_{i \in \Lambda} \ell_\infty(B_{E_i^*}))_{c_0} = (\sum_{i \in \Lambda} \ell_\infty(\Gamma_i))_{c_0}$  be the canonical embedding. For each  $n \in \Lambda$ , let  $Q_n : (\sum_{i \in \Lambda} \ell_\infty(\Gamma_i))_{c_0} \rightarrow \ell_\infty(\Gamma_n)$  be the canonical projection. Let  $P_n : \ell_\infty(\Gamma_n) \rightarrow E_n$  be a family of projections with  $\|P_n\| \leq \lambda$ . For each  $n \in \Lambda$  and  $m \in \Gamma_n$ , let  $e_{nm} \in (\sum_{i \in \Lambda} \ell_\infty(\Gamma_i))_{c_0}$  with the standard biorthogonal functionals  $e_{nm}^* \in ((\sum_{i \in \Lambda} \ell_\infty(\Gamma_i))_{c_0})^*$  such that  $e_{ij}^*(e_{nm}) = \delta_{in} \delta_{jm}$ . Then

$$x = \sum_{n \in \Lambda} \{(e_{nm}^*)(x)\}_{m \in \Gamma_n} \text{ for all } x \in X.$$

By Theorem 1.3, for each  $n \in \Lambda$  and  $m \in \Gamma_n$ , there exists  $\phi_{nm} \in B_{Y^*}$  with  $\|\phi_{nm}\| = \|e_{nm}^*\|$  such that

$$|\langle \phi_{nm}, f(x) \rangle - \langle e_{nm}^*, x \rangle| \leq 4\varepsilon, \text{ for all } x \in X.$$

Clearly,  $e_{nm}^* \rightarrow 0$  uniformly for each  $m \in \Gamma_n$  in the  $w^*$ -topology of  $Z^*$ . Let

$$K = \{\psi \in B(Y^*) : |\langle \psi, f(x) \rangle| \leq 4\varepsilon, \text{ for all } x \in X\}.$$

Since  $\Gamma_n$  can be well ordered for every  $n \in \Lambda$ , we write

$$\Gamma_n = \{0, 1, 2, \dots, w_0, w_0 + 1, \dots, w_1, \dots \prec \Gamma_n\},$$

where  $\Gamma_n$  also denotes by its ordinal number. It follows from i) that for each  $n \in \Lambda$ , there is a net  $(\psi_{n0}) \subset K$  such that  $d(\phi_{n0}, \psi_{n0}) \rightarrow 0$ . We can choose  $(\psi_{nm}) \subset K$  such that for every  $n \in \Lambda$  and  $m \in \Gamma_n$ ,  $d(\phi_{nm}, \psi_{nm}) \leq d(\phi_{n0}, \psi_{n0})$  or equivalently,  $(\phi_{nm} - \psi_{nm}) \rightarrow 0$  uniformly for each  $m \in \Gamma_n$  in the  $w^*$ -topology of  $Y^*$ . Let  $Q : Y \rightarrow (\sum_{i \in \Lambda} \ell_\infty(\Gamma_i))_{c_0}$  be defined for all  $y \in Y$  by

$$Q(y) = \sum_{n \in \Lambda} \{ \langle \phi_{nm} - \psi_{nm}, y \rangle \}_{m \in \Gamma_n} \in (\sum_{i \in \Lambda} \ell_\infty(\Gamma_i))_{c_0},$$

which yields that

$$\|Q(y)\| \leq (\sup_{n \in \Lambda, m \in \Gamma_n} \|\phi_{nm} - \psi_{nm}\|) \|y\| \leq 2\|y\|.$$

Thus

$$\|Q\| \leq 2.$$

Let  $S : Y \rightarrow X$  be defined for all  $y \in Y$  by

$$S(y) = \sum_{n \in \Lambda} P_n Q_n Q(y) = \sum_{n \in \Lambda} P_n \{ \langle \phi_{nm} - \psi_{nm}, y \rangle \}_{m \in \Gamma_n}.$$

Hence

$$\|S\| = \sup_{n \in \Lambda} \|P_n Q_n Q\| \leq 2\lambda$$

and

$$\begin{aligned} \|Sf(x) - x\| &= \left\| \sum_{n \in \Lambda} P_n \{ \langle \phi_{nm} - \psi_{nm}, f(x) \rangle \}_{m \in \Gamma_n} - \sum_{n \in \Lambda} P_n \{ \langle e_{nm}^*, x \rangle \}_{m \in \Gamma_n} \right\| \\ &\leq \lambda \sup_{n \in \Lambda} \sup_{m \in \Gamma_n} | \langle \phi_{nm}, f(x) \rangle - \langle e_{nm}^*, x \rangle - \langle \psi_{nm}, f(x) \rangle | \\ &\leq \lambda (\sup_{n \in \Lambda} \sup_{m \in \Gamma_n} | \langle \phi_{nm}, f(x) \rangle - \langle e_{nm}^*, x \rangle | + \sup_{n \in \Lambda} \sup_{m \in \Gamma_n} | \langle \psi_{nm}, f(x) \rangle |) \\ &\leq 8\varepsilon\lambda. \end{aligned}$$

Thus, our proof is completed.  $\square$

**Remark 4.7.** There are many other examples for separably injective Banach spaces, such as the Johnson-Lindenstrauss spaces [23], Benyamini-space which is an M-space nonisomorphic to a  $C(K)$ -space [6] and the WCG nontrivial twisted sums of  $c_0(\Gamma)$  constructed by Argyros, Castillo, Granero, Jimenez and Moreno [2] (see, for instance, [4]).

Qian [30] proved that the pair  $(L_p, L_p)$  is stable for  $1 < p < \infty$ . Šemrl and Väisälä [32] gave a sharp estimate for the constant pair  $(\alpha, \gamma)$  with  $\gamma = 2$ . Therefore, it is very natural to ask:

**Problem 4.8.** *Is it true that the following pairs are stable for  $1 \leq p \leq \infty$  and  $p \neq q < \infty$ ?*

- (1)  $((\sum_{n=1}^{\infty} l_p^n)_{c_0}, (\sum_{n=1}^{\infty} l_p^n)_{c_0})$ ; (2)  $((\sum_{n=1}^{\infty} l_p^n)_{\ell_{\infty}}, (\sum_{n=1}^{\infty} l_p^n)_{\ell_{\infty}})$ ;
- (3)  $((\sum_{n=1}^{\infty} \ell_{\infty})_{l_p}, (\sum_{n=1}^{\infty} \ell_{\infty})_{l_p})$ ; (4)  $((\sum_{n=1}^{\infty} l_p)_{\ell_{\infty}}, (\sum_{n=1}^{\infty} l_p)_{\ell_{\infty}})$ ;
- (5)  $((\sum_{n=1}^{\infty} L_p)_{\ell_{\infty}}, (\sum_{n=1}^{\infty} L_p)_{\ell_{\infty}})$ ; (6)  $((\sum_{n=1}^{\infty} c_0)_{l_p}, (\sum_{n=1}^{\infty} c_0)_{l_p})$ ;
- (7)  $((\sum_{n=1}^{\infty} L_p)_{c_0}, (\sum_{n=1}^{\infty} L_p)_{c_0})$ ; (8)  $((\sum_{n=1}^{\infty} \ell_p)_{c_0}, (\sum_{n=1}^{\infty} \ell_p)_{c_0})$ .
- (9)  $((\sum_{n=1}^{\infty} l_p)_{\ell_q}, (\sum_{n=1}^{\infty} l_p)_{\ell_q})$ ; (10)  $((\sum_{n=1}^{\infty} L_p)_{\ell_q}, (\sum_{n=1}^{\infty} L_p)_{\ell_q})$ .

It is true for (1), (2), (3), (4) and (5) if  $p = \infty$  as we have proved. In this case, it is not true for (6), (7) and (8) since  $(\sum_{n=1}^{\infty} c_0)_{\ell_{\infty}}$ ,  $(\sum_{n=1}^{\infty} L_{\infty})_{c_0}$  and  $(\sum_{n=1}^{\infty} \ell_{\infty})_{c_0}$  are not complemented in  $\ell_{\infty}$ . If  $1 \leq p < \infty$ , then it is also not true for (3), (4) and (5) since  $(\sum_{n=1}^{\infty} \ell_{\infty})_{l_p}$ ,  $(\sum_{n=1}^{\infty} l_p)_{\ell_{\infty}}$  and  $(\sum_{n=1}^{\infty} L_p)_{\ell_{\infty}}$  are not complemented in  $\ell_{\infty}$ . However, we do not know if it is true or not for the above problem 4.8 in general case.

## 5 Acknowledgements

This work was partially done while the author was visiting Texas A&M University and in Analysis and Probability Workshop at Texas A&M University which was funded by NSF Grant. The author would like to thank Professor W.B. Johnson and Professor Th. Schlumprecht for the invitation. Some contribution to this work (in Section 3) has been done by W.B. Johnson since 2012, and the author would like to thank him for discussion. This work is also a part of the author's Ph.D. thesis under the supervision of Professor Lixin Cheng.

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